THE IDEMPOTENTS IN CYCLOTOMIC HECKE ALGEBRAS AND PERIODIC PROPERTY OF THE JUCYS-MURPHY ELEMENTS

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Abstract. This paper proves a periodic property of Jucys-Murphy elements of the degenerate and non-degenerate cyclotomic Hecke algebras of type A. We do this by first giving a new closed formula for the KLR idempotents $e(\mathbf{i})$ which, it tuns out, is very efficient computationally.

1. Introduction

The degenerate and non-degenerate cyclotomic Hecke algebras H_n^Λ of type A are important algebras because they arise in the categorification of the canonical basis of the affine special linear groups [2]. Recently, building on works of Khovanov and Lauda [4, 5] and Rouquier [9], Brundan and Kleshchev [1] showed that H_n^Λ is isomorphic to a Khovanov-Lauda-Rouquier algebra of type A, and hence that these algebras are $\mathbb{Z}$-graded.

Central to construction of Brundan and Kleshchev graded isomorphism theorem are certain deponents $e(\mathbf{i}) \in H_n^\Lambda$, which in the case of the symmetric groups have their origins in the work of Murphy [8, (1,2)]. The purpose of this paper is to give explicit formulas for these idempotents when $e > 0$ and $p > 0$, which are very efficient computationally, and to use them to prove a periodicity property for the Jucys-Murphy elements of H_n^Λ.

This paper is organized as follows. In Section 2 we introduced the definition of (degenerate and non-degenerate) cyclotomic Hecke algebras of type A. We then give the definition of $e(\mathbf{i})$'s constructed by Kleshchev [6, Lemma 7.1] in H_n^Λ and a set of nilpotency elements $\{\, y_r \mid 1 \le r \le n \,\}$ defined by Brundan and Kleshchev [1, (3.21),(4.21)]. In Section 3 we gave an explicit expression of idempotents $e(\mathbf{i})$. In Section 4 and 5 we proved the periodic property of Jucys-Murphy elements in degenerate and non-degenerate cyclotomic Hecke algebras of type A, respectively. In Section 6 we improved our results of Section 3 and showed that an expression of $e(\mathbf{i})$'s depended on the nilpotency degree of y_r's.

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2. Cyclotomic Hecke algebras of type A and the idempotents $e(\mathbf{i})$

Let $\mathbb{F}_p$ be a fixed field of characteristic $p \ge 0$ with $q \in \mathbb{F}_p^\times$. Let e be the smallest positive integer such that $1 + q + \ldots + q^{e-1} = 0$ and setting $e = 0$ if no such integer exists. Then define $I = \mathbb{Z}/e\mathbb{Z}$ if $e > 0$ and $I = \mathbb{Z}$ if $e = 0$.

For $n \ge 0$, assume that $q = 1$. Let H_n be the **degenerate affine Hecke algebra**, working over $\mathbb{F}_p$. So H_n has generators

$$\{x_1, \ldots, x_n\} \cup \{s_1, \ldots, s_{n-1}\}$$

subject to the following relations

$$\begin{aligned}
x_r x_s &= x_s x_r; \\
s_r x_{r+1} &= x_r s_r + 1, \qquad s_r s_x = x_s s_r \qquad \text{if } s \ne r, r+1; \\
s_r^2 &= 1; \\
s_r s_{r+1} s_r &= s_{r+1} s_r s_{r+1}, \qquad s_r s_t = s_t s_r \qquad \text{if } |r-t| > 1.
\end{aligned}$$

Now we assume that $q \ne 1$ and H_n be the **non-degenerate affine Hecke algebra** over $\mathbb{F}_p$. So H_n has generators

$$\{X_1^{\pm1}, \ldots, X_n^{\pm1}\} \cup \{T_1, \ldots, T_{n-1}\}$$

subject to the following relations

$$\begin{aligned}
X_r^{\pm1} X_s^{\pm1} &= X_s^{\pm1} X_r^{\pm1}, \qquad X_r X_r^{-1} = 1; \\
T_r X_r T_r &= q X_{r+1}, \qquad T_r X_s = X_s T_r \qquad \text{if } s \ne r, r+1; \\
T_r^2 &= (q-1) T_r + q; \\
T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, \qquad T_r T_s = T_s T_r \qquad \text{if } |r-s| > 1.
\end{aligned}$$

To the index set I, we associate two lattices

$$P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i, \qquad Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i,$$

and let $(.,.) : P \times Q \longrightarrow \mathbb{Z}$ be the bilinear pairing defined by $(\Lambda_i, \alpha_j) = \delta_{i,j}$. Let P_+ and Q_+ denote the subset of P and Q such that $P_+ = \bigoplus_{i\in I} \mathbb{N}\Lambda_i$ and $Q_+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i$.

Then for any $\Lambda \in P_+$, we define

$$H_n^\Lambda = \begin{cases} H_n/\langle \prod_{i\in I}(X_1 - q^i)^{(\Lambda,\alpha_i)}\rangle, & \text{if } q \neq 1; \\ H_n/\langle \prod_{i\in I}(x_1 - i)^{(\Lambda,\alpha_i)}\rangle, & \text{if } q = 1. \end{cases} \tag{2.1}$$

and we call H_n^Λ the **degenerate cyclotomic Hecke algebra** if $q = 1$ and **non-degenerate cyclotomic Hecke algebra** if $q \neq 1$. We call p the **characteristic** of H_n^Λ and e the **quantum characteristic** of H_n^Λ.

By the definitions, degenerate and non-degenerate cyclotomic Hecke algebras are similar with some minor difference. In order to minimize their difference we define

$$q_i = \begin{cases} i, & \text{if } q = 1; \\ q^i, & \text{if } q \neq 1. \end{cases} \tag{2.2}$$

and use x_r instead of X_r when we don't have to distinguish which case we are working with. Hence we can rewrite (2.1) as

$$H_n^\Lambda = H_n/\langle \prod_{i\in I}(x_1 - q_i)^{(\Lambda,\alpha_i)}\rangle. \tag{2.3}$$

In H_n^Λ, the elements x_r and X_r are called the **Jucys-Murphy elements**.

We can define a set of pairwise orthogonal idempotents $\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ for both degenerate and non-degenerate H_n^Λ. By Kleshchev [6, Lemma 7.1], the eigenvalues of each x_r or X_r on M belongs to I. So $M = \bigoplus_{\mathbf{i}\in I^n} M_{\mathbf{i}}$ of its weight space

$$M_{\mathbf{i}} = \{\, v \in M \mid (x_r - q_{i_r})^N v = 0 \text{ for all } r = 1, \dots, d \text{ and } N \gg 0 \,\}, \tag{2.4}$$

where q_{i_r} is introduced in (2.2). Then we deduce that there is a system $\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ such that $e(\mathbf{i})M = M_{\mathbf{i}}$, and $\mathbf{i} \in I^n$ is called the residue sequence.

Brundan and Kleshchev [1, (3.21),(4.21)] introduced a set of elements y_r in both degenerate and non-degenerate cyclotomic Hecke algebras, where

$$y_r := \begin{cases} \sum_{\mathbf{i}\in I^n}(x_r - i_r)e(\mathbf{i}), & \text{if } q = 1; \\ \sum_{\mathbf{i}\in I^n}(1 - q^{-i_r}X_r)e(\mathbf{i}). & \text{if } q \neq 1. \end{cases} \tag{2.5}$$

and by [1, Lemma 2.1], in both cases $y_r^s = 0$ for $s \gg 0$. It is easy to imply that $(X_r - q^{i_r})^s e(\mathbf{i}) = 0$ for $s \gg 0$ in non-degenerate case. Hence we have the following Corollary.

Corollary 2.6. *We have $(x_r - q_{i_r})^s e(\mathbf{i}) = 0$ for $s \gg 0$ in both degenerate and non-degenerate cyclotomic Hecke algebras. Furthermore, $(x_r - q_{i_r})^s e(\mathbf{i}) = 0$ if and only if $y_r^s e(\mathbf{i}) = 0$.*

We will apply Corollary 2.6 without mention.

The smallest s with $y_r^s e(\mathbf{i}) = 0$ is called the **nilpotency degree** of y_r corresponds to $e(\mathbf{i})$. Using the explicit expression of $e(\mathbf{i})$ we found, we can find the periodic property of the Jucys-Murphy elements in both degenerate and non-degenerate cyclotomic Hecke algebras when $e > 0$ and $p > 0$, and the period of x_r is determined by the nilpotency degree of y_r's.

3. Explicit expression of idempotents $e(\mathbf{i})$ part I

3.1 General cases

In this section, first we introduce some properties of characteristic and quantum characteristic.

Lemma 3.1. *Suppose $\mathbb{F}_p$ is a field with char $\mathbb{F}_p = p > 0$ and $r_1, r_2 \in H_n^\Lambda$. For any non-negative integer k we always have $(r_1 - r_2)^{p^k} = r_1^{p^k} - r_2^{p^k}$.*

Remark 3.2. Notice the above Lemma is a well-known result and will be applied without mention in this chapter.

Lemma 3.3. *Suppose p and e are characteristic and quantum characteristic of degenerate H_n^Λ. Then $e = p$.*

Proof. The Lemma follows directly by the definitions of p and e in degenerate H_n^Λ. □

Lemma 3.4. *Suppose p and e are characteristic and quantum characteristic of non-degenerate H_n^Λ with $e, p > 0$. Then $gcd(e, p) = 1$. Moreover, we can find l such that $p^l \equiv 1 \pmod{e}$.*

Proof. In non-degenerate case, $\gcd(e, p) = 1$ is well-known. So by Chinese Remainder Theorem we can find $a, b \in \mathbb{Z}$ such that $ap + be = 1$. Now consider the sequence $p, p^2, p^3, p^4 \dots$. We can find k_1, k_2 such that $p^{k_1} \equiv p^{k_2} \pmod e$. Choose k_2 such that $k_2 - k_1 > k_1$. Hence write $l = k_2 - k_1$ and $p^l \equiv s \pmod e$ where $0 \le s \le e - 1$. So we have $p^{2l} \equiv p^l \pmod e$ which implies $s^2 \equiv s \pmod e$. So we can write $s^2 - s = ke$ for some $k \in \mathbb{Z}$. So

$$s^2 - s = ke \quad \Rightarrow \quad as(s-1) = ake \quad \Rightarrow \quad (1 - be)(s-1) = ake \quad \Rightarrow \quad s - 1 = (b(s-1) + ak)e$$

which implies $e \mid s - 1$. But because $0 \le s \le e - 1$, we have $s = 1$. Therefore $p^l \equiv 1 \pmod e$. □

Therefore, in degenerate case, by Lemma 3.3 we have $e = p$ and in non-degenerate case, by Lemma 3.4 we have $\gcd(e, p) = 1$, i.e. $e \neq p$.

In the rest of this paper we set $p > 0$ and $e > 0$. Fix a residue sequence $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$. For any $1 \le r \le n$ and any $j \in I$ with $j \neq i_r$, choose $N \gg 0$ and define $L_{i_r,j} = 1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^N$ in both degenerate and non-degenerate cases.

Notice that by the definition of $e(\mathbf{i})$ given in (2.4), for any $\mathbf{j} \in I$ and $1 \le r \le n$, we have

$$(x_r - q_{j_r})^N e(\mathbf{j}) = 0$$

for $N \gg 0$.

Lemma 3.5. *Suppose $1 \le r \le n$ and $\mathbf{j} = (j_1, j_2, \dots, j_n) \in I^n$. For $j \in I$ and $N_j \gg 0$ we have*

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = \begin{cases} e(\mathbf{j}), & \text{if } j_r = i_r; \\ 0, & \text{if } j_r = j. \end{cases}$$

Proof. Suppose $j_r = i_r$. Because $(x_r - q_{i_r})^N e(\mathbf{j}) = (x_r - q_{j_r})^N e(\mathbf{j}) = 0$ for $N \gg 0$, we have

$$L_{i_r,j} e(\mathbf{j}) = (1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^N) e(\mathbf{j}) = e(\mathbf{j}) - \frac{1}{(q_{i_r} - q_j)^N} (q_{i_r} - x_r)^N e(\mathbf{j}) = e(\mathbf{j}).$$

Therefore $L_{i_r,j}^{N_j} e(\mathbf{j}) = L_{i_r,j}^{N_j - 1} e(\mathbf{j}) = \dots = L_{i_r,j} e(\mathbf{j}) = e(\mathbf{j})$.

Suppose $j_r = j$. We have

$$\begin{aligned}
L_{i_r,j} &= 1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^N \\
&= -\sum_{k=1}^{N} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k + \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\
&= \frac{x_r - q_{i_r}}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k + \frac{q_{i_r} - q_j}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\
&= (x_r - q_{i_r} + q_{i_r} - q_j) \frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\
&= (\frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k)(x_r - q_j).
\end{aligned}$$

Therefore for $N_j \gg 0$,

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = (\frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k)^{N_j} (x_r - q_j)^{N_j} e(\mathbf{j}) = 0$$

because when $j_r = j$ we have $(x_r - q_j)^{N_j} e(\mathbf{j}) = 0$, which completes the proof. □

Now we define $L_r(\mathbf{i}) = \prod_{\substack{j \in I \\ j \neq i_r}} L_{i_r,j}$. In the product, $j \in I \backslash \{i_r\}$, which is a finite product since $e > 0$. So $L_r(\mathbf{i})$ is well defined. We have the following Lemma.

Lemma 3.6. *Suppose $1 \le r \le n$. We can choose $N_r(\mathbf{i}) \gg 0$ such that*

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \sum_{\substack{\mathbf{j} \in I^n \\ j_r = i_r}} e(\mathbf{j}).$$

Proof. By Lemma 3.5, for any $j \in I$ with $j \neq i_r$ we can find N_j large enough such that

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = \begin{cases} e(\mathbf{j}), & \text{if } j_r = i_r, \\ 0, & \text{if } j_r = j. \end{cases}$$

Now choose $N_r(\mathbf{i}) \geq \max\{N_j \mid j \in I, j \neq i_r\}$, which is finite since $e > 0$. Therefore $L_r(\mathbf{i})^{N_r(\mathbf{i})} = \prod_{\substack{j \in I \\ j \neq i_r}} L_{i_r,j}^{N_r(\mathbf{i})}$. Hence for any $e(\mathbf{j})$, if $j_r \neq i_r$,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \prod_{\substack{j \in I \\ j \neq i_r}} L_{i_r,j}^{N_r(\mathbf{i})} e(\mathbf{j}) = \Big(\prod_{\substack{j \in I \\ j \neq i_r, j_r}} L_{i_r,j}^{N_r(\mathbf{i})}\Big) L_{r,j_r}^{N_r(\mathbf{i})} e(\mathbf{j}) = 0, \tag{3.7}$$

and if $j_r = i_r$,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = e(\mathbf{j}), \tag{3.8}$$

because for any j, $L_{i_r,j}^{N_j} e(\mathbf{j}) = e(\mathbf{j})$.

Therefore, because $\sum_{\mathbf{j} \in I^n} e(\mathbf{j}) = 1$, by (3.7) and (3.8),

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = L_r(\mathbf{i})^{N_r(\mathbf{i})} \Big(\sum_{\mathbf{j} \in I^n} e(\mathbf{j})\Big) = \sum_{\mathbf{j} \in I^n} L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \sum_{\substack{\mathbf{j} \in I^n \\ j_r = i_r}} L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \sum_{\substack{\mathbf{j} \in I^n \\ j_r = i_r}} e(\mathbf{j})$$

which completes the proof. □

As the idempotents $e(\mathbf{j})$'s are pairwise orthogonal, Lemma 3.6 immediately implies the following.

Corollary 3.9. *For any $\mathbf{i} \in I^n$, we have*

$$e(\mathbf{i}) = \prod_{r=1}^{n} L_r(\mathbf{i})^{N_r(\mathbf{i})}.$$

The previous results are true in both degenerate and non-degenerate cases. Notice that when we define $L_{i_r,j} = 1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^{N_j}$ and $L_r(\mathbf{i})^{N_r(\mathbf{i})}$, the only restriction is that N_j and $N_r(\mathbf{i})$ are large enough. As we now show, by choosing specific values for N_j and $N_r(\mathbf{i})$, it is possible to simplify the expression of $L_r(\mathbf{i})^{N_r(\mathbf{i})}$ even further and give a more explicit expression of $e(\mathbf{i})$. We emphasize the simplified expressions for $L_r(\mathbf{i})^{N_r(\mathbf{i})}$ are different for degenerate and non-degenerate H_n^Λ.

3.2 Explicit expression in degenerate cases

We start with the degenerate cyclotomic Hecke algebras. Recall that in this case $e = p$.

Proposition 3.10. *Suppose $q = 1$. For any $i_r \in I$ there exists $s \gg 0$ such that*

$$\sum_{\substack{\mathbf{j} \in I^n \\ j_r = i_r}} e(\mathbf{j}) = \begin{cases} 1 - x_r^{p^s(1-p)}, & \textit{when } i_r = 0, \\ -\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k}, & \textit{when } i_r \neq 0. \end{cases}$$

Proof. By Lemma 3.6 the Proposition is equivalent to claim that

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \begin{cases} 1 - x_r^{p^s(1-p)}, & \text{when } i_r = 0, \\ -\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k}, & \text{when } i_r \neq 0. \end{cases}$$

for $s \gg 0$.

By the definition of $L_r(\mathbf{i})$, because $I = \mathbb{Z}/p\mathbb{Z}$ we have

$$L_r(\mathbf{i}) = \prod_{\substack{j \in I \\ j \neq i_r}} L_{i_r,j} = \prod_{\substack{j \in I \\ j \neq i_r}} \Big(1 - \Big(\frac{i_r - x_r}{i_r - j}\Big)^{N_j}\Big) = \prod_{j=1}^{p-1} \Big(1 - \Big(\frac{i_r - x_r}{j}\Big)^{N_j}\Big).$$

Take $k \gg 0$ and $N_j = p^k$. Hence because H_n^Λ is defined over a field $\mathbb{F}_p$ of characteristic p, we have $j^{N_j} = j$. And because p is a prime, we have

$$L_r(\mathbf{i}) = \prod_{j=1}^{p-1} \Big(1 - \Big(\frac{i_r - x_r}{j}\Big)^{N_j}\Big) = \prod_{j=1}^{p-1} \Big(1 - \frac{(i_r - x_r)^{N_j}}{j}\Big) = \prod_{j=1}^{p-1} (1 - j \cdot (i_r - x_r)^{N_j}) = 1 - (i_r - x_r)^{(p-1)N_j}.$$

Without loss of generality, choose $N_r(\mathbf{i}) = p^l$ with $l \gg 0$. We have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = (1 - (i_r - x_r)^{(p-1)N_j})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_j N_r(\mathbf{i})}.$$

Setting $s = k + l$, we have $N_jN_r(\mathbf{i}) = p^{k+l} = p^s$. Now we consider two cases, which are $i_r = 0$ and $i_r \neq 0$. Suppose first $i_r = 0$. We have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} = 1 - (-x_r)^{(p-1)p^s} = 1 - x_r^{(p-1)p^s}. \tag{3.11}$$

Suppose $i_r \neq 0$. We have

$$\begin{aligned}
(i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} &= (i_r - x_r)^{p^{s+1}-p^s} = \frac{(i_r - x_r)^{p^{s+1}}}{(i_r - x_r)^{p^s}} \\
&= \frac{i_r - x_r^{p^{s+1}}}{i_r - x_r^{p^s}} = \frac{1 - (\frac{x_r}{i_r})^{p^{s+1}}}{1 - (\frac{x_r}{i_r})^{p^s}} \\
&= 1 + (\frac{x_r}{i_r})^{p^s} + (\frac{x_r}{i_r})^{2p^s} + \ldots + (\frac{x_r}{i_r})^{(p-1)p^s} \\
&= 1 + \frac{x_r^{p^s}}{i_r} + \frac{x_r^{2p^s}}{i_r^2} + \ldots + \frac{x_r^{(p-1)p^s}}{i_r^{p-1}} = \sum_{k=0}^{p-1} \frac{x_r^{kp^s}}{i_r^k}.
\end{aligned}$$

Hence,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} = 1 - \sum_{k=0}^{p-1} \frac{x^{kp^s}}{i_r^k} = -\sum_{k=1}^{p-1} \frac{x^{kp^s}}{i_r^k}. \tag{3.12}$$

By combining (3.11) and (3.12), we complete the proof. ☐

Finally, by combining Corollary 3.9 and Proposition 3.10, we have an explicit expression of $e(\mathbf{i})$ for the degenerate cyclotomic Hecke algebras.

Theorem 3.13. *Suppose* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ *and* $q = 1$, *then*

$$e(\mathbf{i}) = \prod_{r=1}^{n} L_r(\mathbf{i})^{N_r(\mathbf{i})}$$

where

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \begin{cases} 1 - x_r^{p^s(1-p)}, & \text{when } i_r = 0, \\ -\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k}, & \text{when } i_r \neq 0. \end{cases}$$

for $s \gg 0$.

3.3 Explicit expression in non-degenerate cases

We now give a similar expression for the non-degenerate cyclotomic Hecke algebras. First we give two Lemmas which will be used later.

Lemma 3.14. *For any* $k \in \mathbb{Z}$ *with* $k \not\equiv 0 \pmod{e}$, *we have*

$$1 + q^k + q^{2k} + \ldots + q^{(e-1)k} = 0.$$

Proof. By the definition, we have

$$\begin{aligned}
& 1 + q + q^2 + \ldots + q^{e-1} = 0 \\
\Rightarrow\ & (1 + q + q^2 + \ldots + q^{e-1})(1 - q) = 0 \\
\Rightarrow\ & 1 - q^e = 0 \\
\Rightarrow\ & q^e = 1 \\
\Rightarrow\ & (q^e)^k = q^{ke} = (q^k)^e = 1 \\
\Rightarrow\ & (q^k)^e - 1 = (1 + q^k + q^{2k} + \ldots + q^{(e-1)k})(q^k - 1) = 0.
\end{aligned}$$

Because $k \in \mathbb{Z}$ and $k \not\equiv 0 \pmod{e}$, we have $q^k - 1 \neq 0$. Therefore we must have $1+q^k+q^{2k}+\ldots+q^{(e-1)k} = 0$. ☐

Lemma 3.15. *Suppose* $i_r \in I$ *and* $f(x) = \prod_{j \neq i_r}(1 - \frac{r^{i_r} - x}{r^{i_r} - r^j}) \in \mathbb{F}_p[x]$ *with* $r = q^s$ *for some positive integer* $s \not\equiv 0 \pmod{e}$ *and* $q \in \mathbb{F}_p^\times$. *Then* $e^{-1} \in \mathbb{F}_p$ *and*

$$f(x) = e^{-1}(1 + \frac{x}{r^{i_r}} + (\frac{x}{r^{i_r}})^2 + \ldots + (\frac{x}{r^{i_r}})^{e-1}).$$

Proof. By Lemma 3.4 we have $\gcd(e,p)=1$ and hence $e^{-1}\in\mathbb{F}_p$. Define $g(x)=e^{-1}(1+\frac{x}{r^{i_r}}+(\frac{x}{r^{i_r}})^2+\ldots+(\frac{x}{r^{i_r}})^{e-1})$. We prove that $f(x)=g(x)$ by first comparing their roots. It is obvious that the roots of $f(x)$ are all of the form r^j with $j\in I$ and $j\neq i_r$. Then for any such r^j,

$$g(r^j)=e^{-1}(1+r^{j-i_r}+r^{2(j-i_r)}+\ldots+r^{(e-1)(j-i_r)})=e^{-1}(1+r^k+r^{2k}+\ldots+r^{(e-1)k})$$

for $k\equiv j-i_r \pmod e$ and $k\neq 0$. Because $r=q^s$ and $s\not\equiv 0 \pmod e$, we must have $sk\not\equiv 0 \pmod e$. Therefore by Lemma 3.14 we have $g(r^j)=0$. Because $f(x)$ and $g(x)$ are both polynomials of degree $e-1$, they have $e-1$ roots, which means that $g(x)$ and $f(x)$ have the same roots. This yields that $f(x)=kg(x)$ for some $k\in\mathbb{F}_p$.

Now because $f(r^{i_r})=1=g(r^{i_r})$, we have $k=1$. Therefore $f(x)=g(x)$, which completes the proof. □

Proposition 3.16. *Suppose $q\neq 1$. For any $i_r\in I$, there exists $s\gg 0$ such that*

$$\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j})=e^{-1}(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1}).$$

Proof. By Lemma 3.6 the Proposition is equivalent to prove that

$$L_r(\mathbf{i})^{N_r(\mathbf{i})}=e^{-1}(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1}).$$

By the definition of $L_r(\mathbf{i})$, because $I=\mathbb{Z}/e\mathbb{Z}$, if $N_j, N_r(\mathbf{i})\gg 0$ then we have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})}=\prod_{\substack{j\in I\\ j\neq i_r}}(1-(\frac{q^{i_r}-X_r}{q^{i_r}-q^j})^{N_j})^{N_r(\mathbf{i})}.$$

Suppose $N_j=p^k$ and $N_r(\mathbf{i})=p^l$ with $k,l\gg 0$. We have

$$\begin{aligned}
L_r(\mathbf{i})^{N_r(\mathbf{i})} &= \prod_{j\neq i_r}(1-(\frac{q^{i_r}-X_r}{q^{i_r}-q^j})^{p^k})^{p^l}\\
&= \prod_{j\neq i_r}(1-(\frac{q^{i_r}-X_r}{q^{i_r}-q^j})^{p^{k+l}})\\
&= \prod_{j\neq i_r}(1-\frac{q^{p^{k+l}\cdot i_r}-X_r^{p^{k+l}}}{q^{p^{k+l}\cdot i_r}-q^{p^{k+l}\cdot j}})\\
&= \prod_{j\neq i_r}(1-\frac{r^{i_r}-X_r^{p^s}}{r^{i_r}-r^j}),
\end{aligned}$$

where $s=k+l$ and $r=q^{p^s}\in\mathbb{F}_p$. Notice that by Lemma 3.4, we have $p^s\not\equiv 0 \pmod e$.

Now we set $f(x)=\prod_{j\neq i_r}(1-\frac{r^{i_r}-x}{r^{i_r}-r^j})\in\mathbb{F}_p[x]$. By Lemma 3.15 we have

$$f(x)=e^{-1}(1+\frac{x}{r^{i_r}}+(\frac{x}{r^{i_r}})^2+\ldots+(\frac{x}{r^{i_r}})^{e-1}).$$

Therefore

$$\begin{aligned}
L_r(\mathbf{i})^{N'} &= f(X_r^{p^s})=e^{-1}(1+\frac{X_r^{p^s}}{r^{i_r}}+(\frac{X_r^{p^s}}{r^{i_r}})^2+\ldots+(\frac{X_r^{p^s}}{r^{i_r}})^{e-1})\\
&= e^{-1}(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1})
\end{aligned}$$

which completes the proof. □

Finally we can get an explicit expression of $e(\mathbf{i})$ for the non-degenerate H_n^Λ using Proposition 3.16 and the orthogonality of $e(\mathbf{i})$'s.

Theorem 3.17. *Suppose $\mathbf{i}=(i_1,i_2,\ldots,i_n)\in I^n$ and $q\neq 1$, we have*

$$e(\mathbf{i})=e^{-n}\prod_{r=1}^{n}(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1})$$

for $s\gg 0$.

4. Periodic property of x_r in degenerate case

In the degenerate cyclotomic Hecke algebras, suppose $e = p > 0$ and $H_n^\Lambda(\mathbb{Z}_p)$ is the degenerate cyclotomic Hecke algebra over $\mathbb{Z}_p$. It is obvious that $H_n^\Lambda = H_n^\Lambda(\mathbb{Z}_p)\otimes\mathbb{F}_p$. When $p > 0$ the algebra $H_n^\Lambda(\mathbb{Z}_p)$ is finite. We know that $\dim H_n^\Lambda = \ell^n n!$. Hence over $\mathbb{Z}_p$ the algebra has $p\ell^n n!$ elements. Therefore, by choosing $k > p\ell^n n!$, for any r we must be able to find k_1, k_2 with $1 \le k_1 < k_2 \le k$ such that $x_r^{k_1} = x_r^{k_2}$. Therefore for any r we can find integers d_r and N such that for any $N' \ge N$, $x_r^{N'} = x_r^{N'+d_r}$. Because $H_n^\Lambda = H_n^\Lambda(\mathbb{Z}_p)\otimes\mathbb{F}_p$, we have the same property for x_r's in H_n^Λ. We define the **period** of x_r to be the smallest positive integer d_r such that $x_r^N = x_r^{N+d_r}$ for some N.

In this section we will give information on the d_r and the minimal N such that $x_r^N = x_r^{N+d_r}$ when $e = p > 0$.

4.1 Form of the period d_r

Recall $y_r := \sum_{\mathbf{i}\in I^n}(x_r - i_r)e(\mathbf{i})$ in degenerate cyclotomic Hecke algebras by (2.5).

Lemma 4.1. *Suppose s is an integer. For any r, $x_r^{p^{s+1}} = x_r^{p^s}$ if and only if $y_r^{p^s} = 0$.*

Proof. For any $i \in I$, we have

$$x_r^{p^{s+1}} - x_r^{p^s} = (x_r^{p^{s+1}} - i) - (x_r^{p^s} - i) = (x_r - i)^{p^{s+1}} - (x_r - i)^{p^s}.$$

Suppose $y_r^{p^s} = \sum_{\mathbf{i}\in I^n}(x_r - i_r)^{p^s}e(\mathbf{i}) = 0$. Therefore for any $\mathbf{i}$, $(x_r - i_r)^{p^s}e(\mathbf{i}) = 0$. Then for any $\mathbf{i} \in I^n$ with $i_r = i$ we have

$$(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}}e(\mathbf{i}) - (x_r - i)^{p^s}e(\mathbf{i}) = 0.$$

Then as $\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) = 1$, we have

$$(x_r^{p^{s+1}} - x_r^{p^s}) = \sum_{\mathbf{i}\in I^n}(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = 0,$$

which shows that $x_r^{p^{s+1}} = x_r^{p^s}$.

Suppose $y_r^{p^s} = \sum_{\mathbf{i}\in I^n}(x_r - i_r)^{p^s}e(\mathbf{i}) \ne 0$, we must be able to find a $\mathbf{i} \in I^n$ such that $(x_r - i_r)^{p^s}e(\mathbf{i}) \ne 0$. Assume

$$(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}}e(\mathbf{i}) - (x_r - i)^{p^s}e(\mathbf{i}) = 0,$$

which means that $y_r^{p^{s+1}}e(\mathbf{i}) = y_r^{p^s}e(\mathbf{i}) \ne 0$. Because $p^{s+1} > p^s$ and $y_r^{p^s}e(\mathbf{i}) \ne 0$, we can find k where $y_r^{p^{s+1}+k}e(\mathbf{i}) = 0$ and $y_r^{p^s+k}e(\mathbf{i}) \ne 0$. But $y_r^{p^{s+1}+k}e(\mathbf{i}) = y_r^k y_r^{p^{s+1}}e(\mathbf{i}) = y_r^k y_r^{p^s}e(\mathbf{i}) = y_r^{p^s+k}e(\mathbf{i}) \ne 0$, which leads to contradiction. Therefore we must have $(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}}e(\mathbf{i}) - (x_r - i)^{p^s}e(\mathbf{i}) \ne 0$, which yields that $x_r^{p^{s+1}} \ne x_r^{p^s}$. □

Choose $s \gg 0$ such that $y_r^s = 0$. By Lemma 4.1 we have $x_r^{p^s} = x_r^{p^{s+1}} = x_r^{p^s+(p-1)p^s}$. So the period d_r divides $p^s(p-1)$. Then $d_r = p^m$ or $p^m(p-1)$ with $m \ge 0$.

Lemma 4.2. *Suppose d_r is the period of x_r. Then $(p-1) \mid d_r$.*

Proof. When $p = 2$ there is nothing to prove. Hence we set $p > 2$ so that p is odd. Assume that $d_r = p^m$ for some m. Consider $\lambda = (r-1, 1^{n-r+1})$ and $\mathfrak{t} = \mathfrak{t}^\lambda$. Let $\mathbf{j} = (j_1, j_2, \ldots, j_n) = \mathrm{res}(\mathfrak{t})$, it is easy to see that $j_r = e - 1 = p - 1$. Now $\mathbf{j}$ is a residue sequence so that $e(\mathbf{j}) \ne 0$ by [3, Lemma 4.1(c)]. So we must have $\sum_{\substack{\mathbf{i}\in I^n\\ i_r=p-1}} e(\mathbf{i}) \ne 0$. Choose $s \gg m$. By Proposition 3.10,

$$\begin{aligned} L_r(\mathbf{j})^{N_r(\mathbf{j})} &= -\frac{x_r^{p^s}}{p-1} - \frac{x_r^{2p^s}}{(p-1)^2} - \ldots - \frac{x_r^{(p-1)p^s}}{(p-1)^{p-1}} \\ &= x_r^{p^s} - x_r^{2p^s} + x_r^{3p^s} - \ldots - x_r^{(p-1)p^s}. \end{aligned}$$

By assumption, because $s \gg m$, we have $x_r^{p^s} = x_r^{2p^s} = \ldots = x_r^{(p-1)p^s}$. Therefore

$$L_r(\mathbf{j})^{N_r(\mathbf{j})} = x_r^{p^s} - x_r^{2p^s} + x_r^{3p^s} - \ldots - x_r^{(p-1)p^s} = (1 - 1 + 1 - \ldots - 1)x_r^{p^s} = 0.$$

But by Lemma 3.6 we have $L_r(\mathbf{j})^{N_r(\mathbf{j})} = \sum_{i_r=p-1} e(\mathbf{i}) \ne 0$, which leads to contradiction. Therefore $d_r = p^m(p-1)$ and hence $(p-1)|d_r$. □

4.2 Period of x_r

Now we know that $d_r = p^m(p-1)$ for some m. We can give a more specific value of m. Define l to be the integer such that $y_r^{p^l} = 0$ and $y_r^{p^{l-1}} \neq 0$. First we introduce two Lemmas.

Lemma 4.3. *Suppose $f(x) \in \mathbb{F}_p[x]$, $h \in H_n^{\Lambda}$ and $e(\mathbf{i})h \neq 0$. Then $f(x_r)e(\mathbf{i})h = 0$ only if $f(i_r) = 0$.*

Proof. We prove this Lemma by contradiction. Because $\mathbb{F}_p$ is a field, $f(i_r) = 0$ only if $(x - i_r) \mid f(x)$. Assume $f(x_r)e(\mathbf{i})h = 0$. Suppose $f(i_r) \neq 0$, we can write $f(x) = (x - i_r)g(x) + j$ with $j \neq 0$. Set $s \gg 0$ such that $(x_r - i_r)^{p^s} e(\mathbf{i}) = 0$. Because $f(x_r)e(\mathbf{i})h = 0$, we have

$$f^{p^s}(x_r)e(\mathbf{i})h = ((x_r - i_r)g(x_r) + j)^{p^s} e(\mathbf{i})h = g^{p^s}(x_r)(x_r - i_r)^{p^s} e(\mathbf{i})h + j{\cdot}e(\mathbf{i})h = j{\cdot}e(\mathbf{i})h \neq 0$$

because $j \neq 0$ and $e(\mathbf{i})h \neq 0$, which leads to contradiction. Therefore $f(x_r)e(\mathbf{i})h \neq 0$ when $f(i_r) \neq 0$. This completes the proof. □

Lemma 4.4. *Suppose $k \in \mathbb{Z}$ and $t \in \mathbb{Z}$. For any $i \in I$ with $i \neq 0$, we have*

$$x^k - x^{k+p^t(p-1)} = f(x)(i-x)^{p^t}$$

with $f(x) = \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})$.

Proof. Suppose $i \in I$ and $i \neq 0$. We have

$$\begin{aligned}
x^k - x^{k+p^t(p-1)} &= x^k(1 - x^{p^t(p-1)}) = x^k(1 - (\frac{x^{p^t}}{i})^{p-1}) \\
&= x^k(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(1 - \frac{x^{p^t}}{i}) \\
&= \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(i^{p^t} - x^{p^t}) \\
&= \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(i - x)^{p^t} \\
&= f(x)(i-x)^{p^t}
\end{aligned}$$

with $f(x) = \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})$. This completes the proof. □

Proposition 4.5. *Suppse l is the smallest non-negative integer such that $y_r^{p^l} = 0$. Then the period of x_r is $d_r = p^l(p-1)$.*

Proof. Suppose $d_r = p^m(p-1)$. By Lemma 4.1 we have $x_r^{p^{l+1}} = x_r^{p^l + p^l(p-1)} = x_r^{p^l}$. Therefore $d_r \mid p^l(p-1)$ which indicates that $m \leq l$. Now take $s \gg 0$, by Lemma 4.4 we have

$$(x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^{l-1}}$$

where $f(x) = \frac{x^{p^s}}{i_r}(1 + \frac{x^{p^{l-1}}}{i_r} + (\frac{x^{p^{l-1}}}{i_r})^2 + \ldots + (\frac{x^{p^{l-1}}}{i_r})^{p-2}) \in \mathbb{F}_p[x]$. It is easy to see that $f(i_r) = p - 1 \neq 0$. By the definition of l, $e(\mathbf{i})(i_r - x_r)^{p^{l-1}} \neq 0$. Then by Lemma 4.3 we have

$$(x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^{l-1}} \neq 0.$$

Therefore $x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)} \neq 0$, i.e. $x_r^{p^s} \neq x_r^{p^s + p^{l-1}(p-1)}$, which yields $m \geq l$. This shows that $m = l$ and $d_r = p^l(p-1)$. □

4.3 Periodic property of x_r

Now we know that the period of x_r is $d_r = p^l(p-1)$, and we still need to find the smallest non-negative integer N such that $x_r^N = x_r^{N+d_r}$.

Proposition 4.6. *Suppose $1 \leq r \leq n$ and we can find a residue sequence $\mathbf{i}$ such that $i_r = 0$. If N is the smallest non-negative integer such that $x_r^N \sum_{i_r=0} e(\mathbf{i}) = 0$, then $x_r^N = x_r^{N+d_r}$ and $x_r^{N-1} \neq x_r^{N-1+d_r}$.*

Proof. By the definition of N, we can find $\mathbf{i}$ with $i_r = 0$ such that $x_r^{N-1}e(\mathbf{i}) \neq 0$ and $x_r^N e(\mathbf{i}) = 0$. Suppose $s \gg 0$. Because $d_r \geq 1$ we have

$$(x_r^{N-1} - x_r^{N-1+d_r})e(\mathbf{i}) = x_r^{N-1}e(\mathbf{i}) - x_r^{N-1+d_r}e(\mathbf{i}) = x_r^{N-1}e(\mathbf{i}) \neq 0$$

which indicates that $x_r^{N-1} \neq x_r^{N-1+d_r}$.

Next we will prove that $x_r^N = x_r^{N+d_r}$. Suppose $\mathbf{i} \in I^n$ with $i_r = 0$, then

$$(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = (1 - x_r^{d_r})x_r^N e(\mathbf{i}) = 0$$

by the definition of N. Now suppose $\mathbf{i} \in I^n$ with $i_r \neq 0$. By Proposition 4.5, $d_r = p^l(p-1)$ where $y_r^{p^l} = 0$. So by Lemma 4.4,

$$(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = (x_r^N - x_r^{N+(p-1)p^l})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^l} = f(x_r)e(\mathbf{i})(-y_r)^{p^l} = 0$$

with $f(x) \in \mathbb{F}_p[x]$. Therefore we have $(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = 0$ for any $\mathbf{i} \in I^n$ and hence $x_r^N = x_r^{N+d_r}$. This completes the proof. □

Notice that in Proposition 4.6 we require $1 \leq r \leq n$ such that we can find a residue sequence $\mathbf{i}$ with $i_r = 0$. If no such residue sequence exists we obtain a different result.

Proposition 4.7. *Suppose $1 \leq r \leq n$ and for any residue sequence $\mathbf{i}$ we always have $i_r \neq 0$. Then $x_r^{d_r} = 1$.*

Proof. By Proposition 4.5, $d_r = p^l(p-1)$ where $y_r^{p^l} = 0$. And for any $\mathbf{i} \in I^n$, we have $i_r \neq 0$. Then by Lemma 4.4,

$$(1 - x_r^{d_r})e(\mathbf{i}) = (1 - x_r^{p^l(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^l} = f(x_r)e(\mathbf{i})(-y_r)^{p^l} = 0,$$

which shows that $x_r^{d_r}e(\mathbf{i}) = e(\mathbf{i})$ for any $\mathbf{i} \in I^n$. Hence $x_r^{d_r} = 1$. □

Finally we give the main Theorem of this section by combining Proposition 4.5, Proposition 4.6 and Proposition 4.7.

Theorem 4.8. *In the degenerate cyclotomic Hecke algebras, suppose l is the smallest nonnegative integer such that $y_r^{p^l} = 0$ and N is the smallest nonnegative integer such that $x_r^N \sum_{i_r=0} e(\mathbf{i}) = 0$. Then $x_r^k = x_r^{k+p^m(p-1)}$ if and only if $m \geq l$ and $k \geq N$.*

5. Periodic property of X_r in non-degenerate case

In the non-degenerate cyclotomic Hecke algebras, suppose $H_n^\Lambda(\mathbb{Z}_p[q])$ is the non-degenerate cyclotomic Hecke algebra over $\mathbb{Z}_p[q]$. When $e > 0$ and $p > 0$, $\mathbb{Z}_p[q]$ is finite. Therefore for the same reason as in degenerate cases, we can find N and d_r such that $X_r^N = X_r^{N+d_r}$, i.e. X_r has periodic property in $H_n^\Lambda(\mathbb{Z}_p[q])$. Because $H_n^\Lambda = H_n^\Lambda(\mathbb{Z}_p[q])\otimes\mathbb{F}_p$, we have the same property for X_r's in H_n^Λ. The period d_r in non-degenerate cases is defined similarly as in degenerate cases.

In this section we will give information on the d_r and the minimal N such that $X_r^N = X_r^{N+d_r}$ when $e > 0$ and $p > 0$.

5.1 Form of period d_r

Recall that $y_r = \sum_{\mathbf{i}\in I^n}(1 - q^{-i_r}X_r)e(\mathbf{i})$ in non-degenerate cyclotomic Hecke algebras by (2.5).

Lemma 5.1. *Suppose $s \gg 0$ and $1 \leq r \leq n$. We have $X_r^{ep^s} = 1$.*

Proof. By Proposition 3.16, for any $i_r \in I$, we have

$$\begin{aligned}
(X_r - q^{i_r})^{p^s} \sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}) &= (X_r^{p^s} - q^{p^s \cdot i_r}) \sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}) \\
&= e^{-1}(X_r^{p^s} - q^{p^s\cdot i_r})\left(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \left(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\right)^2 + \ldots + \left(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\right)^{e-1}\right) \\
&= e^{-1}\left(\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}} - q^{p^s\cdot i_r}\right) = 0,
\end{aligned}$$

which leads to

$$\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}} = q^{p^s\cdot i_r} \quad \Rightarrow \quad X_r^{ep^s} = q^{ep^s\cdot i_r} = 1$$

because $q^e = 1$. □

Define d_r to be the period of X_r. By Lemma 5.1 we have $d_r \mid ep^s$. Therefore $d_r = ep^m$ with $m \geq 0$ or $d_r = p^m$ with $m \geq 1$. In the following Lemma we are going to give more information about the form of d_r.

Lemma 5.2. *Suppose d_1 is the period of X_1. We have $d_1 = p^m$ if $\Lambda = \ell\Lambda_0$.*

Proof. By (2.3) we have $(X_1 - q^0)^\ell = (X_1 - 1)^\ell = 0$. Choose s such that $p^s \geq \ell$, we have $(X_1 - 1)^{p^s} = X_1^{p^s} - 1 = 0$, which means $X_1^{p^s} = 1$. Hence $d_1 \mid p^s$ and therefore $d_1 = p^m$. □

Remark 5.3. When we set $r = 1$ and $\Lambda = \ell\Lambda_0$, it means that $e(\mathbf{i}) = 0$ if $i_r = i_1 \neq 0$. So Lemma 5.2 is actually:

Suppose $1 \leq r \leq n$ *and for any* $\mathbf{i} \in I^n$, $e(\mathbf{i}) = 0$ *if* $i_r \neq 0$. *Then* $d_r = p^m$.

because the only possible r and Λ for such condition is giving in Lemma 5.2.

Lemma 5.4. *Suppose* d_r *is the period of* X_r. *We have* $e \mid d_r$ *if* $r > 1$ *or* $r = 1$ *and* $\Lambda \neq \ell\Lambda_0$.

Proof. We prove the Lemma by contradiction. Assume that $d_r = p^m$. Choose $i_r \in I$ with $i_r \neq 0$. Because $r > 1$ or $r = 1$ and $\Lambda \neq \ell\Lambda_0$ we must can find $\mathbf{j} \in I^n$ with $j_r = i_r$ with $e(\mathbf{j}) \neq 0$. Then $\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) \neq 0$. Choose $s \gg m$. By Lemma 3.4, $\gcd(e, p) = 1$. Then because $i_r \neq 0$, $p^s{\cdot}i_r \not\equiv 0 \pmod e$. Then by Lemma 3.14 and Proposition 3.16, we have

$$\begin{aligned}
\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) &= e^{-1}\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big)\\
&= e^{-1}\Big(1 + \frac{1}{q^{p^s\cdot i_r}} + \frac{1}{q^{2p^s\cdot i_r}} + \ldots + \frac{1}{q^{(e-1)p^s\cdot i_r}}\Big)X_r^{p^s}\\
&= \frac{e^{-1}}{q^{(e-1)p^s\cdot i_r}}\big(1 + q^{p^s\cdot i_r} + (q^{p^s\cdot i_r})^2 + \ldots + (q^{p^s\cdot i_r})^{e-1}\big)X_r^{p^s} = 0,
\end{aligned}$$

which leads to contradiction. Hence $d_r \neq p^m$ and therefore $e \mid d_r$. □

5.2 Period of X_r

Now we know that $d_r = p^m$ when $r = 1$ and $\Lambda = \ell\Lambda_0$ and $d_r = ep^m$ otherwise. In the rest of the section we will find the value of m. First we give the simpler case.

Lemma 5.5. *Suppose* $s \geq 0$ *and* $\Lambda = \ell\Lambda_0$. *We have* $X_1^{p^s} = 1$ *if and only if* $y_1^{p^s} = 0$.

Proof. Suppose $y_1^{p^s} = 0$. For any $\mathbf{i} \in I^n$,

$$y_1^{p^s} e(\mathbf{i}) = (1 - X_1)^{p^s} e(\mathbf{i}) = e(\mathbf{i}) - X_1^{p^s} e(\mathbf{i}) = 0 \quad\Rightarrow\quad X_1^{p^s} e(\mathbf{i}) = e(\mathbf{i}).$$

Therefore $X_1^{p^s} = 1$.
Suppose $y_1^{p^s} \neq 0$. Then we can find $\mathbf{i} \in I^n$ with $y_1^{p^s} e(\mathbf{i}) \neq 0$. So

$$y_1^{p^s} e(\mathbf{i}) = (1 - X_1)^{p^s} e(\mathbf{i}) = e(\mathbf{i}) - X_1^{p^s} e(\mathbf{i}) \neq 0 \quad\Rightarrow\quad X_1^{p^s} e(\mathbf{i}) \neq e(\mathbf{i}).$$

Therefore $X_1^{p^s} \neq 1$. □

Now we consider the case when $r \neq 1$ or $r = 1$ and $\Lambda \neq \ell\Lambda_0$.

Lemma 5.6. *For any non-negative integer* s, *we can find* $k \gg s$ *such that* $q^{p^k} = q^{p^s}$ *and* $p^{k-s} \equiv 1 \pmod e$.

Proof. By Lemma 3.4, we can find l such that $p^l \equiv 1 \pmod e$. Because $q^e = 1$, choose $t \gg 0$ and set $k = s + tl$, we have $q^{p^k} = q^{p^{s+tl}} = q^{p^s p^{tl}} = (q^{p^{tl}})^{p^s} = q^{p^s}$. Moreover, $p^{k-s} = p^{tl} \equiv 1^t \pmod e \equiv 1 \pmod e$. This completes the proof. □

Now we are ready to give more information of d_r.

Lemma 5.7. *We have* $X_r^{ep^s} = 1$ *for some* s *only if* $y_r^{p^s} = 0$.

Proof. Fix s such that $X_r^{ep^s} = 1$. By Lemma 5.6, we can find $k \gg 0$ such that $q^{p^{s+k}} = q^{p^s}$ and $p^k \equiv 1 \pmod e$. Therefore $p^{s+k} - p^s = p^s(p^k - 1)$ and hence $ep^s \mid p^s(p^k - 1)$. So $X_r^{p^{s+k}} = X_r^{p^s}$.
Then for any $i_r \in I$, by Proposition 3.16, we have

$$\begin{aligned}
\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) &= e^{-1}\Big(1 + \frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}} + \Big(\frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}}\Big)^{e-1}\Big)\\
&= e^{-1}\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}(X_r-q^{i_r})^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) &= e^{-1}(X_r-q^{i_r})^{p^s}(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1})\\ &= e^{-1}(X_r^{p^s}-q^{p^s\cdot i_r})(1+\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2+\ldots+(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1})\\ &= e^{-1}(\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}}-q^{p^s\cdot i_r}) = e^{-1}(\frac{1}{q^{(e-1)p^s\cdot i_r}}-q^{p^s\cdot i_r}) = 0\end{aligned}$$

because $\frac{1}{q^{(e-1)p^s\cdot i_r}} = q^{p^s\cdot i_r}$. This means that $y_r^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) = (1-q^{-i_r}X_r)^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) = 0$ for any $i_r\in I$. Hence $y_r^{p^s}=0$. □

Lemma 5.8. *Suppose $y_r^{p^s}=0$ for some s. Then we have $X_r^{ep^s}=1$.*

Proof. Fix s such that $y_r^{p^s}=0$. For any $\mathbf{i}\in I^n$, by Lemma 5.6 we can choose $k\gg s$ such that $q^{p^k}=q^{p^s}$. Then

$$\begin{aligned}(X_r^{p^k}-X_r^{p^s})e(\mathbf{i}) &= (X_r^{p^k}-q^{p^s\cdot i_r}-X_r^{p^s}+q^{p^s\cdot i_r})e(\mathbf{i})\\ &= (X_r^{p^k}-q^{p^k\cdot i_r})e(\mathbf{i})-(X_r^{p^s}-q^{p^s\cdot i_r})e(\mathbf{i})\\ &= (X_r-q^{i_r})^{p^k}e(\mathbf{i})-(X_r-q^{i_r})^{p^s}e(\mathbf{i}) = q^{p^k\cdot i_r}(-y_r)^{p^k}e(\mathbf{i})-q^{p^s\cdot i_r}(-y_r)^{p^s}e(\mathbf{i})=0.\end{aligned}$$

So $(X_r^{p^k}-X_r^{p^s})e(\mathbf{i})=0$ for any $\mathbf{i}\in I^m$. Therefore we must have $X_r^{p^k}-X_r^{p^s}=0$ for some $k\gg 0$. Hence

$$X_r^{p^k}-X_r^{p^s}=X_r^{p^s}(X_r^{p^k-p^s}-1)=0 \quad\Rightarrow\quad X_r^{p^k-p^s}=1,$$

which implies that $d_r\mid p^k-p^s$. We know that $d_r=ep^m$ for some m and $p^k-p^s=p^s(p^{k-s}-1)$. It is obvious that $m\le s$. Hence $X_r^{ep^s}=1$. □

The next Corollary follows straightforward by combining Lemma 5.7 and Lemma 5.8.

Corollary 5.9. *Suppose $s\ge 0$, $r>1$ or $r=1$ and $\Lambda\neq\ell\Lambda_0$. We have $X_r^{ep^s}=1$ if and only if $y_r^{p^s}=0$.*

Finally, combining all the results above, we have the final Theorem.

Theorem 5.10. *In non-degenerate H_n^Λ, we have $X_r^{d_r}=1$ with*

$$d_r=\begin{cases}p^m, & \text{if } r=1 \text{ and } \Lambda=\ell\Lambda_0;\\ ep^m, & \text{otherwise.}\end{cases}$$

if and only if $y_r^{p^m}=0$.

Proof. The Theorem follows straightforward by Lemma 5.2, Lemma 5.4, Lemma 5.5 and Corollary 5.9. □

6. Explicit expression of idempotents $e(\mathbf{i})$ part II

In Theorem 3.13 and Theorem 3.17, we gave explicit expressions of $e(\mathbf{i})$'s in degenerate and non-degenerate cyclotomic Hecke algebras. But the expression depends on the choice of s, and up to now the only information we know about s is that $s\gg 0$. In this section we give a more specific value for s in both degenerate and non-degenerate case.

Define l_r to be the smallest integer such that $y_r^{p^{l_r}}=0$.

Theorem 6.1. *Suppose $\mathbf{i}=(i_1,i_2,\ldots,i_n)\in I^n$ and $q=1$. Then $e(\mathbf{i})=\prod_{r=1}^n L_r(\mathbf{i})^{N_r(\mathbf{i})}$, where*

$$L_r(\mathbf{i})^{N_r(\mathbf{i})}=\begin{cases}1-x_r^{p^{l_r}(1-p)}, & \text{when } i_r=0;\\ -\sum_{k=1}^{p-1}\frac{x_r^{kp^{l_r}}}{i_r^k}, & \text{when } i_r\neq 0.\end{cases}$$

Proof. By Theorem 3.13, we have that $e(\mathbf{i})=\prod_{r=1}^n L_r(\mathbf{i})^{N_r(\mathbf{i})}$, where

$$L_r(\mathbf{i})^{N_r(\mathbf{i})}=\begin{cases}1-x_r^{p^s(1-p)}, & \text{when } i_r=0;\\ -\sum_{k=1}^{p-1}\frac{x_r^{kp^s}}{i_r^k}, & \text{when } i_r\neq 0.\end{cases}$$

for $s \gg 0$. Then by Lemma 4.1, it is easy to see that when $s \gg 0$, $x_r^{p^s} = x_r^{p^{l_r}}$, and hence

$$1 - x_r^{p^s(1-p)} = 1 - x_r^{p^{l_r}(1-p)};$$
$$-\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k} = -\sum_{k=1}^{p-1} \frac{x_r^{kp^{l_r}}}{i_r^k},$$

which completes the proof. □

Theorem 6.2. *Suppose* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ *and* $q \neq 1$, *we have*

$$e(\mathbf{i}) = e^{-n} \prod_{r=1}^{n} (1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}).$$

Proof. Suppose $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ and $q \neq 1$. By Theorem 3.17, we have

$$e(\mathbf{i}) = e^{-n} \prod_{r=1}^{n} (1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1})$$

for $s \gg 0$. If $\Lambda = \ell\Lambda_0$, for any $\mathbf{i} \in I^n$ with $e(\mathbf{i}) \neq 0$, by [3, Lemma 4.1(c)] we must have $i_1 = 0$.

Then we have

$$e(\mathbf{i}) = e^{-n} \prod_{r=2}^{n} (1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1})$$

with $s \gg 0$. By Lemma 5.6, choose $s \gg l_r$ such that $q^{p^s} = q^{p^{l_r}}$ and $p^{s-l_r} \equiv 1 \pmod{e}$. Because $p^s - p^{l_r} = (p^{s-l_r} - 1)p^{l_r}$, $ep^{l_r} \mid (p^s - p^l)$. Therefore by Theorem 5.10, for any $r \geq 2$, $X_r^{p^s} = X_r^{p^{l_r} + p^s - p^{l_r}} = X_r^{p^{l_r}}$. This means that

$$1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1} = 1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}$$

for any $r \geq 2$. Hence

$$e(\mathbf{i}) = e^{-n} \prod_{r=2}^{n} (1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}).$$

Now recall $y_r^{p^l} = 0$, we have $X_r^{p^l} e(\mathbf{i}) = 0$ because $i_1 = 0$. Therefore

$$e(\mathbf{i})(1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}) = e(\mathbf{i}),$$

and hence

$$e(\mathbf{i}) = e^{-n} \prod_{r=1}^{n} (1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}).$$

If $\Lambda \neq \ell\Lambda_0$, by Lemma 5.6, choose $s \gg l$ such that $q^{p^s} = q^{p^l}$ and $p^{s-l} \equiv 1 \pmod{e}$. Hence for the same reason as when $\Lambda = \ell\Lambda_0$, we have

$$1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1} = 1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}$$

for any r. Therefore

$$e(\mathbf{i}) = e^{-n} \prod_{r=1}^{n} (1 + \frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}} + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^{l_r}}}{q^{p^{l_r} \cdot i_r}})^{e-1}).$$

□

We remark that we do not know explicit expressions for the $e(\mathbf{i})$'s and in general the x_r and X_r's do not have a periodic property when $p = 0$ or $e = 0$. The main problem is because we do not have analogue result of Lemma 3.1 when $p = 0$ and Lemma 3.15 also fails when $e = 0$. But Lemma 3.5 is still true when $p = 0$ or $e = 0$. Define $I_r(\mathbf{i}) := \{ j \in I \mid j \neq i_r$, there exists $e(\mathbf{j}) \neq 0$ with $j_r = j \}$ and set $L'_r(\mathbf{i}) = \prod_{j \in I_r(\mathbf{i})} L_{i_r,j}$. By [3, Lemma 4.1(c)] we have $\{ e(\mathbf{i}) \mid e(\mathbf{i}) \neq 0 \}$ is finite and hence $L'_r(\mathbf{i})$ is well-defined. Then $L'_r(\mathbf{i})$ satisfy Lemma 3.6 and Corollary 3.9. Hence even though we cannot give a simplified expression of $e(\mathbf{i})$'s similar to Theorem 6.1 and Theorem 6.2 when $p = 0$ or $e = 0$, we can still have an expression for $e(\mathbf{i})$'s which is useful for computation purposes.

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